Friedrichs Extensions of Schrödinger Operators with Singular Potentials

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Abstract

The Friedrichs extension for the generalized spiked harmonic oscillator given by the singular differential operator $-d^2/dx^2 + Bx^2 + Ax^{-2} + \lambda x^{-\alpha}$ $(B > 0, A \geq 0)$ in $L_2(0, \infty)$ is studied. We look at two different domains of definition for each of these differential operators in $L_2(0, \infty)$, namely $C_0^\infty(0, \infty)$ and $D(T_{2,F}) \cap D(M_{\lambda,\alpha})$, where the latter is a subspace of the Sobolev space $W_{2,2}(0, \infty)$. Adjoints of these differential operators on $C_0^\infty(0, \infty)$ exist as result of the null-space properties of functionals. For the other domain, convolutions and Jensen and Minkowski integral inequalities, density of $C_0^\infty(0, \infty)$ in $D(T_{2,F}) \cap D(M_{\lambda,\alpha})$ in $L_2(0, \infty)$ lead to the other adjoints. Further density properties $C_0^\infty(0, \infty)$ in $D(T_{2,F}) \cap D(M_{\lambda,\alpha})$ yield the Friedrichs extension of these differential operators with domains of definition $D(T_{2,F}) \cap D(M_{\lambda,\alpha})$.

Keywords Generalized spiked harmonic oscillators, singular potentials, Friedrichs extension, self-adjoint extension, Jensen inequality, Minkowski inequality.

1. Introduction

We derive the Friedrichs extension of the singular Hamiltonians (differential operator of the generalized spiked harmonic oscillator [1-2])

$$H_\lambda = -d^2/dx^2 + Bx^2 + Ax^{-2} + \lambda x^{-\alpha} \ (B > 0, A \geq 0) \tag{1}$$

in the Hilbert space $L_2(0, \infty)$. H_λ (1) has been studied as a generalization of the spiked harmonic oscillator [3-9]

$$H_0 = -d^2/dx^2 + Bx^2 + l(l+1)x^{-2} + \lambda x^{-\alpha} \ (\ l \ \text{is the angular momentum number}). \tag{2}$$

The generalization (1) lies in A ranging over $[0, \infty)$ instead of $l(l+1)$, $l \in \mathcal{N}$. Although the Friedrichs extension of the conventional spiked harmonic oscillator $(A = 0)$ has many times [4-5] been said to exist (the theory of semi-bounded operators [10, Sec. 5.4]), we go much further in the present article; we construct the generalized spiked harmonic oscillator's Friedrichs extension for all $A \in [0, \infty)$.

The Friedrichs extension derives its significance from the following. To provide the spectral decomposition of a symmetric operator H defined *in* a Hilbert space $\mathcal{H}$, whose domain of definition $D(H)$ lies dense in $\mathcal{H}$, namely $H = \int\limits_{-\infty}^{\infty} \mu d_\mu P_\mu$, where P_μ is an ascending μ-parameter family of projection operators on $\mathcal{H}$ satisfying $P_\mu \to 0$ or I in the strong sense, according as $\mu \to -\infty$ or $+\infty$, the operator H must be self adjoint. Furthermore the projection operators P_μ must satisfy $P_{\mu+0} = P_\mu$ in the strong sense ($P_\lambda \to P_\mu$ as $\lambda \downarrow \mu$) [10, p 181, Theorem 7.17], [13, Theorem on p. 320], and the operator H arises out of the inner-product relation $< Hf \mid g > = \int_{-\infty}^{\infty} \mu d_\mu < P_\mu f \mid g >$ for all $f \in D(H)$ and $g \in \mathcal{H}$. Usually only a symmetric operator is given, as is the case of our Schrödinger operator H_λ. A self-adjoint extension of H must be found in order that the spectral decomposition of H be applicable. For the case of our semi-bounded Schrödinger operator H_λ, the Friedrichs extension provides this self-adjoint extension on account of the semi-boundedness of H_λ.

There are two principal approaches to the investigation of the domain problem of the Hamiltonian (2). The first is to regard the singular term $x^{-\alpha}$ as a perturbation of the well known harmonic oscillator Hamiltonian, the second is to look upon the entire potential as a perturbation of the second order differential operator. To our knowledge, the second approach was never discussed in the literature, while the first approach has been investigated by B. Simon [4] and DeFacio et al [5] and was extensively utilized by Harrell [8]. The investigation of the generalized spiked harmonic oscillator's Friedrichs extension in the Hilbert space $L_2(0, \infty)$ is accomplished by giving a suitable domain of definition as well as the action of this operator upon elements of this domain of definition. Because we want to maintain the symmetry of this operator in $L_2(0, \infty)$, we must choose as domain a dense linear subspace of $L_2(0, \infty)$, and also, at the same time, guarantee a relatively easy transfer of the operator from the left side of the inner product to the right. As it turns out, it is this transfer of the operator within the inner product that causes difficulty, and shall require the elaboration and derivation of some properties of infinitely differentiable function on $(0, \infty)$ *vis-a-vis* derivatives of $L_2(0, \infty)$-functions.

2. Analysis of the Hamiltonian H_λ

Because of the presence of the second derivative operator in the generalized spiked harmonic oscillator (1.1), we note that the second derivative, as an operator in the Hilbert space $L_2(0,\infty)$, induces the minimal and maximal second derivative operators $T_{2,0}$ and T_2 with domains of definition $D(T_{2,0}) = C_0^\infty(0,\infty)$ and $D(T_2) = W_{2,2}(0,\infty)$ respectively. $C_0^\infty(0,\infty)$ is the set of infinitely differentiable complex valued functions on $(0,\infty)$ with compact support, whereas $W_{2,2}(0,\infty)$ designates the Sobolev space consisting of all functions $f \in L_2(0,\infty)$ with $f' \in A(0,\infty)$ and $f'' \in L_2(0,\infty)$, where $A(0,\infty)$ is the space of absolutely continuous complex valued functions on $(0,\infty)$. The respective actions of these two second derivative operators in the Hilbert space $L_2(0,\infty)$ are: $T_{2,0}f = \tau_2 f \equiv -f''$ and $T_2 f = \tau_2 f$ according as $f \in D(T_{2,0})$ or $f \in D(T_2)$, where $\tau_2 = -d^2/dx^2$ acts in the sense of $-(f')'$ almost everywhere on $(0,\infty)$. We moreover have the intermediate situation of the Friedrichs extension $T_{2,F}$ of $T_{2,0}$ with domain of definition and action

$$D(T_{2,F}) = \{f \in W_{2,2}(0,\infty) : f(0) = f'(0) = 0\}, \quad T_{2,F}f = -f'' \quad \forall \ f \in D(T_{2,F}), \qquad (1)$$

which is a self-adjoint extension [10, p 157, Theorem 6.32] of $T_{2,0}$ in $L_2(0,\infty)$.

By means of the non-negative parameter λ and the second derivative operators $T_{2,0}$ and $T_{2,F}$, the perturbed Hamiltonian operator

$$-f''(x) + [Bx^2 + Ax^{-2} + \lambda x^{-\alpha}]f(x), \text{ where } f \in D \ (D \text{ to be specified}),$$

may be looked upon as an operator in the Hilbert space $L_2(0,\infty)$ in two different ways. First as the operator sum $\mathbf{H}_\lambda$ of the two operators $T_{2,0}$ and $M_{\lambda;\alpha}$, and second also as an operator sum H_λ, but of $T_{2,F}$ and $M_{\lambda;\alpha}$. At this point we must stress that $M_{\lambda;\alpha}$ stands for the maximal multiplication operator determined by the positive continuous (measurable) function $M_{\lambda;\alpha}(x) \equiv Bx^2 + Ax^{-2} + \lambda x^{-\alpha}$ with domain of definition and action given by

$$D(M_{\lambda;\alpha}) \equiv \{f \in L_2(0,\infty) : M_{\lambda;\alpha}f \in L_2(0,\infty)\} \ \text{ and } \ (M_{\lambda;\alpha}f) \equiv M_{\lambda;\alpha}f \quad \forall \ f \in D(M_{\lambda;\alpha}) \quad (2)$$

respectively, where $M_{\lambda;\alpha}f$ is the conventional product of the functions $M_{\lambda;\alpha}$ and f. Thus the operators $\mathbf{H}_\lambda$ and H_λ have domains of definition and respective actions given by

$$D(\mathbf{H}_\lambda) \equiv D(T_{2,0}) \cap D(M_{\lambda;\alpha}) = C_0^\infty(0,\infty), \quad \mathbf{H}_\lambda f \equiv -f'' + M_{\lambda;\alpha}f \quad \forall \ f \in D(\mathbf{H}_\lambda); \qquad (3a)$$

$$D(H_\lambda) \equiv D(T_{2,F}) \cap D(M_{\lambda;\alpha}) = \{f \in L_2(0,\infty) : f' \in A(0,\infty); f'', M_{\lambda;\alpha}f \in L_2(0,\infty)\},$$
$$H_\lambda f \equiv -f'' + M_{\lambda;\alpha}f \quad \forall \ f \in D(H_\lambda). \qquad (3b)$$

By denoting the inner product of two elements f and g in the Hilbert space $L_2(0,\infty)$ by $<f \mid g> \equiv \int_0^\infty f(x)\overline{g(x)}dx$, we have, that $\mathbf{H}_\lambda$ and H_λ are symmetric as well as semi-bounded from below, since both $D(\mathbf{H}_\lambda)$ and $D(H_\lambda)$ lie dense in $L_2(0,\infty)$. These facts are direct consequences of: $D(H_\lambda) \supset C_0^\infty(0,\infty)$, $C_0^\infty(0,\infty)$ is dense in $L_2(0,\infty)$,

$$<H_\lambda f \mid g> = <-f'' \mid g> + <M_{\lambda;\alpha}f \mid g> = <f' \mid g'> + <M_{\lambda;\alpha}f \mid g> = <f \mid H_\lambda g> \qquad (4)$$

and

$$<\mathbf{H}_\lambda f \mid g> = < - f'' \mid g> + <M_{\lambda;\alpha} f \mid g> = <f' \mid g'> + <M_{\lambda;\alpha} f \mid g> = <f \mid \mathbf{H}_\lambda g> \qquad (5)$$

for all $f, g \in D(H_\lambda)$ and $D(\mathbf{H}_\lambda)$ respectively. In equations (4) and (5), the last two equalities are an immediate result of: integration by parts; $C_0^\infty(0,\infty)$-functions always have compact subsets of $(0,\infty)$ for supports, as well as the property that $f(0^+)$ and $f'(0^+)$ always exist [10, p 153, Theorem 6.27] with $f(\infty) = f'(\infty) = 0$ for any $f \in W_{2,2}(0,\infty)$. We further point out that the maximal multiplication operator $M_{\lambda;\alpha}$ is the linear combination of the maximal multiplication operators M and $(\cdot)^\alpha$ determined by the continuous (measurable) functions $M(x) = Bx^2 + Ax^{-2}$ and $(\cdot)^\alpha(x) = x^{-\alpha}$ on $(0,\infty)$ respectively. Their corresponding domains of definition are

$$D(M) = \{f \in L_2(0,\infty) : Mf \in L_2(0,\infty)\} \text{ and } D((\cdot)^\alpha) = \{f \in L_2(0,\infty) : (\cdot)^\alpha f \in L_2(0,\infty)\} \quad (6)$$

with self-evident actions in terms of the conventional multiplication of functions. If we define the unperturbed Hamiltonians $\mathbf{H}_0 \equiv T_{2,0} + M$ and $H_0 \equiv T_{2,F} + M$ in the sense of operator addition in the Hilbert space $L_2(0,\infty)$, then the perturbed Hamiltionians assume the forms

$$\mathbf{H}_\lambda = \mathbf{H}_0 + \lambda(\cdot)^\alpha \text{ and } H_\lambda = H_0 + \lambda(\cdot)^\alpha \qquad (7)$$

of the sum two operators in $L_2(0,\infty)$ in terms of the perturbation parameter λ.

3. The Adjoint of $\mathbf{H}_\lambda$

We now consider the adjoints of the λ-parameter family of Hamiltonians $\mathbf{H}_\lambda$ in the Hilbert space $L_2(0,\infty)$, whose domains of definition and accompanying actions are

$$D(\mathbf{H}_\lambda{}^\dagger) \equiv \{g \in L_2(0,\infty) : \exists \; \hat{g} \ni <\mathbf{H}_\lambda f \mid g> = <f \mid \hat{g}> \forall f \in D(\mathbf{H}_\lambda)\} \text{ and } \mathbf{H}_\lambda{}^\dagger g = \hat{g}, \quad (1)$$

where the linear manifold $D(\mathbf{H}_\lambda)$ is replaceable by $C_0^\infty(0,\infty)$ in consequence of $D(\mathbf{H}_\lambda) = C_0^\infty(0,\infty)$. For $g \in D(\mathbf{H}_\lambda{}^\dagger)$ and $f \in C_0^\infty(0,\infty)$ we have that

$$<\mathbf{H}_\lambda f \mid g> = < - f'' \mid g> + <M_{\lambda;\alpha} f \mid g> = <f \mid \mathbf{H}_\lambda{}^\dagger g>, \qquad (2)$$

and because $\mathbf{H}_\lambda{}^\dagger g - M_{\lambda;\alpha} \in L_2^{loc}(0,\infty)$, we may rewrite this equation as

$$< - f'' \mid g> = <f \mid \mathbf{H}_\lambda{}^\dagger g - M_{\lambda;\alpha} g> = \int_0^\infty f(x)\overline{[(\mathbf{H}_\lambda{}^\dagger g)(x) - (M_{\lambda;\alpha} g)(x)]}dx. \qquad (3)$$

Let c be a fixed positive number, then the function

$$h_\lambda(x) = - \int_c^x \int_c^{x_1} (\mathbf{H}_\lambda{}^\dagger g - M_{\lambda;\alpha} g)(x_2)dx_2 dx_1 = - \int_c^x (x - x_2)(\mathbf{H}_\lambda{}^\dagger g - M_{\lambda;\alpha} g)(x_2)dx_2 \quad (4)$$

is continuously differentiable with absolutely continuous h_λ' on $(0,\infty)$ - i. e. $h_\lambda' \in A(0,\infty)$ - and

$$(\tau_2 h_\lambda)(x) = (\mathbf{H}_\lambda{}^\dagger g - M_{\lambda;\alpha} g)(x) \text{ for almost all } x \in (0,\infty). \qquad (5)$$

Moreover, we point to the fact that

$$<f \mid \mathbf{H}_\lambda{}^\dagger g - M_{\lambda;\alpha}g> = <f \mid \tau_2 h_\lambda> = <f' \mid h'_\lambda> = <-f'' \mid h_\lambda> \ \forall f \in C_0^\infty(0,\infty), \qquad (6)$$

whence $<-f'' \mid g> = <-f'' \mid h_\lambda>$ or $<-f'' \mid g-h_\lambda> = 0$ for all $f \in C_0^\infty(0,\infty)$. Therefore, $<\cdot \mid g-h_\lambda>$ determines a linear functional

$$L : C_0^\infty(0,\infty) \mapsto \mathcal{C} \text{ with } L(f) = <f \mid g-h_\lambda> = \int_0^\infty f(x)\overline{[g(x) - h_\lambda(x)]}dx, \qquad (7)$$

whose null space satisfies, in terms of the elementary linear functionals $L_j : C_0^\infty(0,\infty) \mapsto \mathcal{C}$ with $L_j(f)$ defined [10, p 154, Theorem 6.28] as the integral of the function $x^j f(x)$ over the interval $(0,\infty)$ $(j = 0,1)$,

$$N(L) \supset Range(T_{2,0}) = \{f \in C_0^\infty(0,\infty) : L_j(f) = 0 \text{ for } j = 0,1\} = N(L_0) \cap N(L_1). \qquad (8)$$

However, $N(L_0) \cap N(L_1) \subset N(L)$ entails that [10, p 56, Theorem 4.1]

$$L = \overline{c_0}L_0 + \overline{c_1}L_1 \text{ or } L(f) = <f \mid g-h_\lambda> = \int_0^\infty f(x)\overline{[c_0 + c_1 x]}dx \ \forall f \in C_0^\infty(0,\infty), \qquad (9)$$

namely $(g - h_\lambda)(x) = c_0 + c_1 x$ a.e. on $(0,\infty)$ or in terms of the previously defined h_λ

$$g(x) = -\int_c^x \int_c^{x_1} (\mathbf{H}_\lambda{}^\dagger g - M_{\lambda;\alpha}g)(x_2)dx_2 dx_1 + c_0 + c_1 x \ \forall x \in (0,\infty). \qquad (10)$$

Thus g' is absolutely continuous on $(0,\infty)$, $-g'' = h''_\lambda = \mathbf{H}_\lambda{}^\dagger g - M_{\lambda;\alpha}g$ as well as $g'' \in L_2^{loc}(0,\infty)$; in particular $-g'' + M_{\lambda;\alpha}g = \mathbf{H}_\lambda{}^\dagger g$. We have therefore

LEMMA 3.1: The adjoint $\mathbf{H}_\lambda{}^\dagger$ of $\mathbf{H}_\lambda$ in the Hilbert space $L_2(0,\infty)$ has domain of definition and action given by

$$D(\mathbf{H}_\lambda{}^\dagger) = \left\{g \in L_2(0,\infty) : g' \in A(0,\infty), \int_0^\infty |(-g'' + M_{\lambda;\alpha}g)(x)|^2 dx < \infty\right\}$$

$$\text{and } \mathbf{H}_\lambda{}^\dagger g = (-g'' + M_{\lambda;\alpha}g). \qquad (11)$$

PROOF. $M_{\lambda,\alpha}(x)$ continuous on $(0,\infty)$ and $(-g'' + M_{\lambda;\alpha}g) \in L_2(0,\infty)$ implies that $g'' \in L_2^{loc}(0,\infty)$; therefore, $g'' \in L_2^{loc}(0,\infty)$ is omitted from within curly bracket. On the other hand, if g belongs to the set within the curly brackets, then g belongs to the domain $D(\mathbf{H}_\lambda{}^\dagger)$ of $\mathbf{H}_\lambda{}^\dagger$. This is evident from taking inner products with $C_0^\infty(0,\infty)$-functions. □

We note that $(-g'' + M_{\lambda;\alpha}g)$ must be taken collectively, and not as the sum $-g'' + M_{\lambda;\alpha}g$ of two $L_2(0,\infty)$-functions arising out of the operator sum of τ_2 and $M_{\lambda;\alpha}$. The adjoint $\mathbf{H}_\lambda{}^\dagger \supset \mathbf{H}_\lambda$, but $\mathbf{H}_\lambda{}^\dagger \neq \mathbf{H}_\lambda$, because for some $g \in D(\mathbf{H}_\lambda{}^\dagger)$, g'' and $M_{\lambda;\alpha}g$ fail to belong to $L_2(0,\infty)$, but their difference $(-g'' + M_{\lambda;\alpha}g) \in L_2(0,\infty)$, as the following counter-example indicates. We define the function $\psi(x) \equiv x^{1/2-\sqrt{1/4+A}}\phi(x)$ $(0 < A < 3/2)$, where ϕ is the restriction of a $C_0^\infty(\mathcal{R})$-function Φ to $(0,\infty)$, such that $\Phi(x) = 1$ in a neighborhood of the origin.

$$(M_{\lambda;\alpha}\psi)(x) = [Bx^{5/2-\sqrt{1/4+A}} + Ax^{-3/2-\sqrt{1/4+A}} + \lambda x^{1/2-\sqrt{1/4+A}-\alpha}]\phi(x) \quad (0 < \alpha < \sqrt{1/4 + A})$$

fails to be an $L_2(0,\infty)$-function; further,

$$\psi''(x) = -Ax^{-3/2-\sqrt{1/4+A}}\phi(x) + (1-\sqrt{1+4A})x^{-1/2-\sqrt{1/4+A}}\phi'(x) + x^{1/2-\sqrt{1/4+A}}\phi''(x), \quad (12)$$

where $\phi'(x) = \phi''(x) = 0$ in a neighborhood about the origin. $\psi'' \notin L_2(0,\infty)$; however,

$$
\begin{aligned}
(\mathbf{H}_\lambda{}^\dagger\psi)(x) &= -\psi''(x) + (M_{\lambda;\alpha}\psi)(x) = (\sqrt{1+4A}-1)x^{-1/2-\sqrt{1/4+A}}\phi'(x) \\
&+ x^{1/2-\sqrt{1/4+A}}\phi''(x) + [Bx^{5/2-\sqrt{1/4+A}} + \lambda x^{1/2-\sqrt{1/4+A}-\alpha}]\phi(x)
\end{aligned}
\tag{13}
$$

belongs to $L_2(0,\infty)$ (the singularities at 0 cancel as result of the presence of ϕ' and ϕ''.

4. Special Density Properties of $C_0^\infty(0,\infty)$

Since the operator $\mathbf{H}_\lambda$ in $L_2(0,\infty)$ with $D(\mathbf{H}_\lambda) = C_0^\infty(0,\infty)$ and $\mathbf{H}_\lambda f = T_{2,0}f + M_{\lambda,\alpha}f$ has the Hilbert space adjoint $\mathbf{H}_\lambda{}^\dagger$ as defined in Lemma 3.1, it is also possible to ascertain the adjoint $H_\lambda{}^\dagger$ of the operator H_λ with $D(H_\lambda) \equiv D(T_{2,F}) \cap D(M_{\lambda,\alpha})$ and $H_\lambda f \equiv (T_{2,F} + M_{\lambda,\alpha})f$ in terms of the sum of the two operators $T_{2,F}$ and $M_{\lambda,\alpha}$ in $L_2(0,\infty)$. Further, the transfer of $H_\lambda^\dagger$ from the right side of $< f \mid H_\lambda^\dagger g >$ to the left in $< H_\lambda f \mid g >$ for all $f \in D(H_\lambda)$ and $g \in D(H_\lambda{}^\dagger)$ requires integration by parts, which worked so nicely for $< \mathbf{H}_\lambda f \mid g > = < f \mid \mathbf{H}_\lambda{}^\dagger g >$, causes insurmountable difficulties in the case of $f \in D(H_\lambda)$ and $g \in D(H_\lambda{}^\dagger)$. This is because $g \in D(H_\lambda{}^\dagger)$ guarantees merely $g'' \in L_2^{loc}(0,\infty)$, and thus an *a priori* statement concerning $g(0)$ and $g'(0)$ is not possible. Therefore, we shall proceeded in another way.

We note that every $f \in D(T_{2,F})$ is continuous on $[0,\infty)$ and has absolutely continuous f' on $[0,\infty)$ with $f(0) = f'(0) = 0$; hence, a shift of function values by $a > 0$ to the right, yields a $D(T_{2,F})$-function again. Thus the function F_a

$$F_a(x) \equiv 0 \text{ for } 0 \le x < a \text{ and } \equiv f(x-a) \text{ for } x \ge a \; \forall f \in D(T_{2,F}). \tag{1}$$

obtained from f belongs to $D(T_{2,F})$. Out of $F_a \in L_2(0,\infty)$, $F_a'(x) = 0$ for $0 \le x < a$ and $= f'(x-a)$ for $x \ge a$, $F_a''(x) = 0$ for $0 < a$ and $= f''(x-a)$ for $x > a$ readily lead to the properties: $F_a(0) = F_a'(0) = 0$, $F_a \in L_2(0,\infty)$, and F_a' is absolutely continuous with $F_a'' \in L_2(0,\infty)$. This triplet of functions $\{F_a, F_a', F_a''\}$ satisfies [11, p 182, Theorem 9.5]:

$$\|f^{(j)} - F_a{}^{(j)}\| \to 0 \text{ as } a \to 0^+ \; (j = 0,1,2), \tag{2}$$

where these three functions are extendable to all of $\mathcal{R}$ by $F_a(x) \equiv 0 \; (x < 0)$. Each of this triplet we "smoothen" by convoluting them with the $C_0^\infty(\mathcal{R})$-function

$$\delta_\eta(x) \equiv A_\eta exp((x^2 - \eta^2)^{-1}) \text{ for } |x| < \eta \text{ and } \equiv 0 \text{ for } |x| \ge a, \text{ where } \int_{-\infty}^{\infty} \delta_\eta(s)ds = 1 \tag{3}$$

for values of $\eta < a/8$. At this stage we point out that $\delta_\eta(x)dx$ defines a positive measure on $(-\eta,\eta)$ whose measure of the interval $(-\eta,\eta)$ is 1; this permits the utilization of the Jensen integral inequality [11, p 62, Theorem 3.3]. In consequence of $\delta_\eta(\pm\eta) = \delta_\eta'(\pm\eta) = 0$, these "smoothened" functions satisfy [12, p 120, theorem 14]:

$$(F_a * \delta_\eta)' = F_a * \delta_\eta' = F_a' * \delta_\eta \text{ and } (F_a * \delta_\eta)'' = F_a * \delta_\eta'' = F_a'' * \delta_\eta.$$

Turning to the first three derivatives of F_a, we have from

$$|F_a^{(j)}(x) - (F_a^{(j)} * \delta_\eta)(x)| \le \int_{-\eta}^{\eta} |F_a^{(j)}(x) - F_a^{(j)}(x-t)|\delta_\eta(t)dt \ (\eta < a/8; j = 0,1,2) \qquad (4)$$

and Jensen's integral inequality (u^2 being convex on $\mathcal{R}$) that

$$|F_a^{(j)}(x) - (F_a^{(j)} * \delta_\eta)(x)|^2 \le \int_{-\eta}^{\eta} |F_a^{(j)}(x) - F_a^{(j)}(x-t)|^2 \delta_\eta(t)dt \ (j = 0,1,2). \qquad (5)$$

The Minkowski integral inequality [11, p 182, Prob. 19] applied to inequality (5) yields:

$$\|F_a^{(j)} - (F_a^{(j)} * \delta_\eta)\| \le \int_{-\eta}^{\eta} \|F_a^{(j)} - F_{a+t}^{(j)}\|\delta_\eta(t)dt \to 0 \text{ for } \eta \to 0^+ \ (j = 0,1,2), \qquad (6)$$

because each of the maps $t \mapsto F_t^{(j)}$ constitutes [11, p 182, Theorem 9.5] a continuous map $(0,\infty) \mapsto L_2(0,\infty)$ ($j = 0,1,2$). Herefrom the triplet of functions $\{(F_a * \delta_\eta), (F_a * \delta_\eta)', (F_a * \delta_\eta)''\}$ satisfies

$$\|F_a^{(j)} - (F_a^{(j)} * \delta_\eta)\| \to 0 \text{ for } \eta \to 0^+ \ (j = 0,1,2). \qquad (7)$$

We further proceed by introducing the $C_0^\infty(\mathcal{R})$-function $\chi_{[-R-r,R+r]} * \delta_r$ for $R, r > 0$, where $\chi_{[-R-r,R+r]} * \delta_r$ assumes on $[-R,R]$ and $\mathcal{R} \setminus [-R-2r, R+2r]$ the values of 1 and 0 respectively, and maps each of the intervals $[-R-2r, -R)$ and $(R, R+2r]$ monotonely onto $[0,1)$. Let $\varphi_{R,r}$ be the restriction of $\chi_{[-R-r,R+r]} * \delta_r$ to $[0,\infty)$. Thus $\varphi_{R,r}([0,R]) = \{1\}$, $\varphi_{R,r}((R, R+2r]) = [0,1)$, $\varphi_{R,r}([R+2r, \infty)) = \{0\}$, and $\varphi_{R,r}'(x) = -\delta_r(x - R - r)$ is a $C_0^\infty(0,\infty)$-function with $supp(\varphi_{R,r}') = [R, R+2r]$. Its effect on the triplet $\{(F_a * \delta_\eta), (F_a * \delta_\eta)', (F_a * \delta_\eta)''\}$ is as follows. First, $(F_a * \delta_\eta)\varphi_{R,r}$ is a $C_0^\infty(0,\infty)$-function, for which

$$\|F_a * \delta_\eta - (F_a * \delta_\eta)\varphi_{R,r}\| = \|(F_a * \delta_\eta)[1 - \varphi_{R,r}]\| \le \|(F_a * \delta_\eta)\chi_{[R,\infty)}\| \to 0 \text{ for } R \to \infty \qquad (8)$$

implies

$$\|F_a * \delta_\eta - (F_a * \delta_\eta)\varphi_{R,r}\| \to 0 \text{ for } R \to \infty. \qquad (9)$$

Second, the chain of inequalities

$$\|(F_a * \delta_\eta)' - ((F_a * \delta_\eta)\varphi_{R,r})'\| = \|(F_a' * \delta_\eta)[1 - \varphi_{R,r}] - (F_a * \delta_\eta)\varphi_{R,r}'\| \le \|(F_a' * \delta_\eta)[1 - \varphi_{R,r}]\|$$
$$+ \|(F_a * \delta_\eta)\varphi_{R,r}'\| \le \|(F_a' * \delta_\eta)\chi_{[R,\infty)}\| + \|\delta_\eta\|_\infty\|(F_a * \delta_\eta)\chi_{[R,r+2r]}\| \to 0 \text{ for } R \to \infty$$
$$(10)$$

entails that

$$\|(F_a * \delta_\eta)' - ((F_a * \delta_\eta)\varphi_{R,r})'\| \to 0 \text{ for } R \to \infty. \qquad (11)$$

And thirdly, by estimating as follows

$$\|(F_a * \delta_\eta)'' - ((F_a * \delta_\eta)\varphi_{R,r})''\| = \|F_a'' * \delta_\eta - (F_a'' * \delta_\eta)\varphi_{R,r} - 2(F_a' * \delta_\eta)\varphi_{R,r}' - (F_a * \delta_\eta)\varphi_{R,r}''\|$$
$$\le \|(F_a'' * \delta_\eta)[1 - \varphi_{R,r}]\| + 2\|(F_a' * \delta_\eta)\varphi_{R,r}'\| + \|(F_a * \delta_\eta)\varphi_{R,r}''\|$$
$$\le \|(F_a'' * \delta_\eta)\chi_{[R,\infty)}\| + 2\|\delta_\eta\|_\infty\|(F_a' * \delta_\eta)\chi_{[R,r+2r]}\| + \|\delta_\eta'\|_\infty\|(F_a * \delta_\eta)\chi_{[R,r+2r]}\|$$
$$(12)$$

we arrive at

$$\|(F_a * \delta_\eta)'' - ((F_a * \delta_\eta)\varphi_{R,r})''\| \to 0 \quad \text{for} \quad R \to \infty. \qquad (13)$$

In summary we have that the triplet of functions $\{(F_a * \delta_\eta)\varphi_{R,r}, ((F_a * \delta_\eta)\varphi_{R,r})', ((F_a * \delta_\eta)\varphi_{R,r})''\}$ satisfies

$$\|(F_a * \delta_\eta)^{(j)} - ((F_a * \delta_\eta)\varphi_{R,r})^{(j)}\| \to 0 \quad \text{for} \quad R \to \infty \ (j = 0, 1, 2). \tag{14}$$

Let us now turn to the maximal multiplication operator $M_{\lambda,\alpha}$ in $L_2(0, \infty)$ by deriving more general properties of the function F_a constructed from $f \in D(H_\lambda)$. Specifically, let $\mu(x)$ be non-negative continuous function on $(0, \infty)$, which is strictly decreasing and strictly increasing on $(0, r_\mu]$ and $[R_\mu, \infty)$ respectively with $r_\mu \leq R_\mu$, and in addition satisfies

$$\lim_{x \to \infty} \frac{\mu(x+1)}{\mu(x)} = C_\mu < \infty. \tag{15}$$

LEMMA 4.1: $\sqrt{\mu}f \in L_2(0, \infty)$ implies that $\sqrt{\mu}F_a \in L_2(0, \infty)$ for all $a \geq 0$.

PROOF. The validity of this statement follows from first trapping a between two successive non-negative integers, namely $n \leq a < n+1$, and estimating thereafter as follows:

$$\int_0^\infty |F_a(x)|^2 \mu(x) dx = \int_0^\infty |f(x)|^2 \mu(x+a) dx \leq \int_0^{R'} |f(x)|^2 \mu(x+a) dx + \int_{R'}^\infty |f(x)|^2 \mu(x+n+1) dx$$

$$= \int_0^{R'} |f(x)|^2 \mu(x+a) dx + \int_{R'}^\infty |f(x)|^2 \left[\frac{\mu(x+n+1)}{\mu(x)} \right] \mu(x) dx$$

$$= \int_0^{R'} |f(x)|^2 \mu(x+a) dx + \int_{R'}^\infty |f(x)|^2 \left\{ \prod_{k=0}^n \frac{\mu(x+k+1)}{\mu(x+k)} \right\} \mu(x) dx$$

$$\leq \int_0^{R'} |f(x)|^2 \mu(x+a) dx + [1 + C_\mu]^{n+1} \int_{R'}^\infty |f(x)|^2 \mu(x) dx < \infty, \tag{16}$$

where $R' > R_\mu$ and is picked so large that $\frac{\mu(x+1)}{\mu(x)} \leq 1 + C_\mu$ for all $x \geq R'$. $\square$

Estimating $\|\sqrt{\mu}f - \sqrt{\mu}F_a\|$, for sufficiently small a, leads to the following

LEMMA 4.2:
For $f \in D(T_{2,F})$ satisfying $\sqrt{\mu}f \in L_2(0, \infty)$, we have that $\|\sqrt{\mu}f - \sqrt{\mu}F_a\| \to 0$ as $a \to 0^+$.

PROOF. We make a restriction on a, namely $0 < a < \min\{r_\mu, 1\}$, and in terms of a positive $r < r_\mu$, which remains unspecified at this moment, we estimate as follows

$$\|\sqrt{\mu}f - \sqrt{\mu}F_a\| = \|[\sqrt{\mu}f - \sqrt{\mu}F_a]\chi_{(0,r]}\| + \|[\sqrt{\mu}f - \sqrt{\mu}F_a]\chi_{(r,\infty)}\|$$

$$\leq \|(\sqrt{\mu}f)\chi_{(0,r]}\| + \|(\sqrt{\mu}F_a)\chi_{(0,r]}\| + \|\sqrt{\mu}(f - F_a)\chi_{(r,\infty)}\|$$

$$\leq \|(\sqrt{\mu}f)\chi_{(0,r]}\| + \|(\sqrt{\mu}F_a)\chi_{(0,r]}\| + \|\sqrt{\mu}(f - F_a)\chi_{(r,b+1]}\| + \|\sqrt{\mu}(f - F_a)\chi_{(b+1,\infty)}\|$$

$$\leq \|(\sqrt{\mu}f)\chi_{(0,r]}\| + \|(\sqrt{\mu}F_a)\chi_{(0,r]}\| + \|\sqrt{\mu}(f - F_a)\chi_{(r,b+1]}\| + \|(\sqrt{\mu}f)\chi_{(b+1,\infty)}\| + \|(F_a)\chi_{(b+1,\infty)}\| \tag{17}$$

We further estimate the last norm expression, by replacing the variable of integration x by $x+1$ under the condition $0 < a < \min\{r_\mu, 1\}$, and lets us arrive at

$$\|(\sqrt{\mu}F_a)\chi_{[b+1,\infty)}\|^2 = \int_{b+1-a}^\infty |f(x)|^2 \mu(x+a) dx \leq \int_b^\infty |f(x)|^2 \mu(x+1) dx$$

$$= \int_b^\infty |f(x)|^2 [\mu(x+1)/\mu(x)] \mu(x) dx \leq [1 + C_\mu] \int_b^\infty |f(x)|^2 \mu(x) dx \tag{18}$$

where $(b \geq R')$ with R' the same as in Lemma 4.1. We thus have for the previous norm estimate in terms of the immediately preceeding integral estimate that

$$
\begin{aligned}
\|\sqrt{\mu}f - \sqrt{\mu}F_a\| &\leq \sqrt{\int_0^r |f(x)|^2 \mu(x) dx} + \sqrt{\int_0^r |F_a(x)|^2 \mu(x) dx} \\
&+ \sqrt{\int_r^{b+1} |f(x) - f(x-a)|^2 \mu(x) dx} + \left[1 + \sqrt{1 + C_\mu}\right] \sqrt{\int_b^\infty |f(x)|^2 \mu(x) dx} \quad (b \geq R').
\end{aligned}
\tag{19}
$$

Now we turn to choosing a sufficiently small so that $\|\sqrt{\mu}f - \sqrt{\mu}F_a\| < \epsilon$. Because $\|\sqrt{\mu}f\| < \infty$, we can deduce the existence of $r(\epsilon) \leq r_\mu$ and $R(\epsilon)$ such that

$$
r < r(\epsilon) \text{ and } \rho > R(\epsilon) \Rightarrow \sqrt{\int_0^r |f(x)|^2 \mu(x) dx}, \; \left[1 + \sqrt{1 + C_\mu}\right] \sqrt{\int_\rho^\infty |f(x)|^2 \mu(x) dx} < \epsilon/4, \tag{20}
$$

whereby the first and fourth integral expressions in inequality (19) are simultaneously less than $\epsilon/4$ provided $r < r(\epsilon)$ and $b > R(\epsilon)$.

If $r < r(\epsilon)$, then the second integral expression in inequality (19) is less than $\epsilon/4$, as can be easily seen from the following calculations. This second integral expression is zero if $r \leq a$ and thus

$$
\begin{aligned}
\sqrt{\int_0^r |F_a(x)|^2 \mu(x) dx} &= \chi_{[0,r)}(a) \sqrt{\int_a^r |f(x-a)|^2 \mu(x) dx} = \chi_{[0,r)}(a) \sqrt{\int_0^{r-a} |f(x)|^2 \mu(x+a) dx} \\
&\leq \chi_{[0,r)}(a) \sqrt{\int_0^{r-a} |f(x)|^2 \mu(x) dx} \leq \chi_{[0,r)}(a) \sqrt{\int_0^r |f(x)|^2 \mu(x) dx} \leq \chi_{[0,r)}(a) \epsilon/4,
\end{aligned}
\tag{21}
$$

because $r < r(\epsilon)$ implies $r \leq r_\mu$ (choice of $r(\epsilon)$) and $\mu(x)$ decreasing on $(0, r_\mu]$ leads to $\mu(x+a) \leq \mu(x)$ since $x + a \leq r - a + a = r$.

By fixing an $r < r(\epsilon)$ and a $b > max\{R', R(\epsilon)\}$, we guarantee that the sum of the first, second, and fourth integral expressions in inequality (19) is less than $3\epsilon/4$. In the third integral expression in inequality (19) f is continuous on $[0, \infty)$, hence uniformly continuous on the closed interval $[0, b+1]$. Therefore, there exists a $\Delta(\epsilon)$ such that

$$
x, y \in [0, b+1] \text{ with } |x - y| < \Delta(\epsilon) \Rightarrow |f(x) - f(y)| < \epsilon/4. \tag{22}
$$

Setting $\mu_b \equiv \max\{\mu(x) : r \geq x \leq b+1\}$, which quantity is non-negative, and thereafter choosing $a < \Delta(\epsilon[4^2(b+1)(\mu_b + 1)]^{-1})$, leads us to

$$
\sqrt{\int_r^{b+1} |f(x) - f(x-a)|^2 \mu(x) dx} < \epsilon/4; \tag{23}
$$

whereby the proof is complete. $\square$

LEMMA 4.3: For $f \in D(T_{2,F})$ satisfying $\sqrt{\mu}f \in L_2(0, \infty)$, we have for each $a > 0$ that $\|\sqrt{\mu}F_a - \sqrt{\mu}F_{a+t}\| \to 0$ as $t \to 0$.

PROOF. Since Lemma 4.2 maintains that $\|\sqrt{\mu}f - \sqrt{\mu}F_a\| \to 0$ as $a \to 0^+$, we are tempted to conclude from Lemma 4.2 the validity of the assertion for $t \to 0$. Unfortunately, we may only conclude the validity for $t \to 0^+$ on account of $\sqrt{\mu}F_a \in L_2(0, \infty)$.

Consequently, we start afresh. The positive quantity a we hold fixed, and trap it between two successive integers, as in the proof of Lemma 4.1, namely $n \leq a < n+1$, and only admit t satisfying $|t| \leq \min\{1, a/4\}$; thus, $3a/4 \leq a+t < n+2$ and $F_a(x) = F_a(x-t)$ for $0 \leq x \leq a/2$. We write

$$\|\sqrt{\mu}F_a - \sqrt{\mu}F_{a+t}\| = \|\sqrt{\mu}(F_a - F_{a+t})\chi_{[a/2, R+2]}\| + \|\sqrt{\mu}(F_a - F_{a+t})\chi_{(R+2, \infty]}\|$$
$$\leq \|\sqrt{\mu}(F_a - F_{a+t})\chi_{[a/2, R+2]}\| + \|(\sqrt{\mu}F_a)\chi_{(R+2, \infty]}\| + \|(\sqrt{\mu}F_{a+t})\chi_{(R+2, \infty]}\| \tag{24}$$

and estimate the last two norm expressions as follows:

$$\|(\sqrt{\mu}F_a)\chi_{(R+2, \infty]}\| = \left[\int_{R+2}^{\infty} |f(x-a)|^2 \mu(x) dx\right]^{1/2} = \left[\int_{R+2-a}^{\infty} |f(x)|^2 \mu(x+a) dx\right]^{1/2}$$
$$\leq \left[\int_{R+2-a}^{\infty} |f(x)|^2 \mu(x+n+2) dx\right]^{1/2} = \left[\int_{R+2-a}^{\infty} |f(x)|^2 \left\{\prod_{k=0}^{n+1} \frac{\mu(x+k+1)}{\mu(x+k)}\right\} \mu(x) dx\right]^{1/2} \tag{25}$$
$$\leq (1+C_\mu)^{1+n/2}\left[\int_{R+2-a}^{\infty} |f(x)|^2 \mu(x) dx\right]^{1/2} \leq (1+C_\mu)^{1+n/2}\|(\sqrt{\mu}f)\chi_{(R+1-a, \infty]}\|$$

for all $R > R'$ and similarly for the other

$$\|(\sqrt{\mu}F_{a+t})\chi_{(R+2, \infty]}\| = \left[\int_{R+2}^{\infty} |f(x-a-t)|^2 \mu(x) dx\right]^{1/2} = \left[\int_{R+2-a-t}^{\infty} |f(x)|^2 \mu(x+a+t) dx\right]^{1/2}$$
$$\leq \left[\int_{R+1-a}^{\infty} |f(x)|^2 \mu(x+n+2) dx\right]^{1/2} = \left[\int_{R+1-a}^{\infty} |f(x)|^2 \left\{\prod_{k=0}^{n+1} \frac{\mu(x+k+1)}{\mu(x+k)}\right\} \mu(x) dx\right]^{1/2}$$
$$\leq (1+C_\mu)^{1+n/2}\left[\int_{R+1-a}^{\infty} |f(x)|^2 \mu(x) dx\right]^{1/2} \leq (1+C_\mu)^{1+n/2}\|(\sqrt{\mu}f)\chi_{(R+1-a, \infty]}\| \tag{26}$$

for all $R > R'$. Thus we have that

$$\|\sqrt{\mu}F_a - \sqrt{\mu}F_{a+t}\| \leq \left[\int_{a/2}^{R+2} |F_a(x) - F_a(x-t)|^2 \mu(x) dx\right]^{1/2} + 2(1+C_\mu)^{1+n/2}\|(\sqrt{\mu}f)\chi_{(R+1-a, \infty]}\| \tag{27}$$

for all $R > R'$ and in consequence of $\|\sqrt{\mu}f\| < \infty$ we can always find an $R''(\epsilon)$ such that

$$(1+C_\mu)^{1+n/2}\|(\sqrt{\mu}f)\chi_{[R, \infty]}\| < \epsilon/4 \text{ for all } R > \max\{R''(\epsilon) + a - 1, R'\}. \tag{28}$$

To estimate the norm expression $\|\sqrt{\mu}(F_a - F_{a+t})\chi_{[a/2, R+2]}\|$ in the intial norm inequality (24) of this proof, we convert this to an integral expression and observe that F_a is continuous on $[0, \infty)$. As a result, F_a is uniformly continuous on the compact interval $[0, R+3]$. This guarantees that to every $\epsilon > 0$ there corresponds a $\Delta_a(\epsilon)$ such that

$$x, y \in [0, R+3] \text{ with } |x-y| < \Delta_a(\epsilon) \Rightarrow |F_a(x) - F_a(y)| < \epsilon/2. \tag{29}$$

By defining the non-negative number $\mu_{a,R} \equiv \max\{\mu(x) : a/2 \leq x \leq R+3\}$, we shall have under the condition $|t| < \min\{\Delta_a(\epsilon^2[4(R+2)(\mu_{a,R}+1)]^{-1}), a/4\}$ that

$$\left[\int_{a/2}^{R+2} |F_a(x) - F_a(x-t)|^2 \mu(x) dx\right]^{1/2} < \epsilon/2. \tag{30}$$

Choosing first in inequality (28) an $R > \max\{R''(\epsilon)+a-1, R'\}$ and thereafter calculating the corresponding $\mu_{a,R}$, we have for the sum of the norms

$$||\sqrt{\mu}(F_a - F_{a+t})\chi_{[a/2,R+2]}|| + ||(\sqrt{\mu}F_a)\chi_{(R+2,\infty)}|| + ||(\sqrt{\mu}F_{a+t})\chi_{(R+2,\infty)}|| < \epsilon, \qquad (31)$$

whereby the proof is complete. $\qquad \square$

LEMMA 4.4: For $f \in D(T_{2,F})$ satisfying $\sqrt{\mu}f \in L_2(0,\infty)$, we have for each $a > 0$ that $||\sqrt{\mu}(F_a - F_a * \delta_\eta)|| \to 0$ as $\eta \to 0^+$.

PROOF. We shall only admit $\eta < a/8$ and return to inequality (4), apply to it Jensen's integral inequality (convexity of the function u^2) which yields for us

$$|F_a(x) - (F_a * \delta_\eta)(x)|^2\mu(x) \le \int_{-\eta}^{\eta} |F_a(x) - F_a(x-t)|^2\mu(x)\delta_\eta(t)dt, \qquad (32)$$

and thus leads us, by means of the Minkowski integral inequality, to

$$\left[\int_0^\infty |F_a(x) - (F_a * \delta_\eta)(x)|^2\mu(x)dx\right]^{1/2} \le \int_{-\eta}^{\eta}\left[\int_0^\infty |F_a(x) - F_a(x-t)|^2\mu(x)dx\right]^{1/2}\delta_\eta(t)dt. \quad (33)$$

We convert both of the expressions in inequality (33), by bringing $\sqrt{\mu(x)}$ underneath the absolute values raised to the power 2, to the simple norm expression

$$||\sqrt{\mu}(F_a - F_a * \delta_\eta)|| \le \int_{-\eta}^{\eta} ||\sqrt{\mu}(F_a - F_{a+t})||\delta_\eta(t)dt. \qquad (34)$$

Because by Lemma 4.3, $||\sqrt{\mu}(F_a - F_{a+t})|| \to 0$ for $t \to 0$ for $a > 0$, we are guaranteed the existence of a $\delta''(\epsilon)$ such that from $|t| < \delta''(\epsilon)$ it follows $||F_a - F_{a+t}|| < \epsilon$. This entails, by the immediately preceeding inequality (34), that $||\sqrt{\mu}(F_a - F_a * \delta_\eta)|| < \epsilon$ for $\eta < \delta''(\epsilon)$; thereby completing the proof. $\qquad \square$

LEMMA 4.5: For $f \in D(T_{2,F})$ with $\sqrt{\mu}f \in L_2(0,\infty)$, $a > 0$ and $0 < \eta < a/8$, we have that $||\sqrt{\mu}(F_a * \delta_\eta - (F_a * \delta_\eta)\varphi_{R,r})|| \to 0$ as $R \to \infty$.

PROOF. We need only to observe that in

$$||\sqrt{\mu}(F_a * \delta_\eta - (F_a * \delta_\eta)\varphi_{R,r})|| = ||\sqrt{\mu}(F_a * \delta_\eta)[1 - \varphi_{R,r}]|| \le ||\sqrt{\mu}(F_a * \delta_\eta)\chi_{[R,\infty)}|| \qquad (35)$$

where the last norm expression tends towards zero as $R \to \infty$. $\qquad \square$

We now have all the tools necessary for ascertaining the adjoint $H_\lambda{}^\dagger$ of the operator H_λ in $L_2(0,\infty)$ in terms of $D(H_\lambda{}^\dagger)$, as well as its action $H_\lambda{}^\dagger : D(H_\lambda{}^\dagger) \to L_2(0,\infty)$.

5. The adjoint of H_λ

We already have the adjoint $\mathbf{H}_\lambda{}^\dagger$ of $\mathbf{H}_\lambda$ in the Hilbert space $L_2(0,\infty)$, because the validity of the inner product equation $< \mathbf{H}_\lambda f \mid g > = < f \mid \mathbf{H}_\lambda{}^\dagger g >$ was easy to establish for all $f \in D(\mathbf{H}_\lambda)$ and $g \in D(\mathbf{H}_\lambda{}^\dagger)$. This was so, because $D(\mathbf{H}_\lambda) = C_0^\infty(0,\infty) \cap D(M_{\lambda,\alpha})$ and $C_0^\infty(0,\infty)$ permitted a direct way of dealing with the integration by parts formula (supports of $C_0^\infty(0,\infty)$-functions are compact subsets of $(0,\infty)$) in spite of the fact that $g \in D(\mathbf{H}_\lambda{}^\dagger)$ guarantees merely that $g'' \in L_2^{loc}(0,\infty)$ and $g' \in A(0,\infty)$ without any indication as to whether $g'(0^+)$ even exists.

In consequence of the previous section dealing with special density properties of $C_0^\infty(0,\infty)$, in particular the three triplet statements (4.2), (4.7) and (4.14), the four lemmas, Lemma 4.2, 4.3, 4.4, 4.5, provide the desired capability of approximating $D(H_\lambda)$-functions by means $C_0^\infty(0,\infty)$-functions. Hence, we shall proceed to determine the adjoint $H_\lambda{}^\dagger$ of H_λ, which process depends upon

THEOREM 5.1: To every $f \in D(H_\lambda) = D(T_{2,F}) \cap D(M_{\lambda,\alpha})$ and $\epsilon > 0$ there exists an $f_\epsilon \in C_0^\infty(0,\infty)$ such that

$$||M_{\lambda,\alpha}[f - f_\epsilon]||, \ ||f^{(j)} - f_\epsilon{}^{(j)}|| < \epsilon \ (j = 0, 1, 2). \tag{1}$$

PROOF. We first note that the positive continuous function $M_{\lambda,\alpha}(x) = [bx^2 + Ax^{-2} + \lambda x^{-\alpha}]$, defining the maximal multiplication operator $M_{\lambda,\alpha}$ in the Hilbert space $L_2(0,\infty)$, is strictly decreasing and strictly increasing on the intervals $(0, x_0]$ and $[x_0, \infty)$ respectively, where $x_0 = x_0(A, B, \alpha; \lambda)$. Futhermore, $M_{\lambda,\alpha}(x)/M_{\lambda,\alpha}(x+1) \to 1$ as $x \to \infty$ from elementary limit calculation. Therefore we conclude that $\mu(x) \equiv M_{\lambda,\alpha}{}^2(x) = [bx^2 + Ax^{-2} + \lambda x^{-\alpha}]^2$ is an admissible function for each of the Lemmas 4.2, 4.4 and 4.5 of section 4. Therefore, we are justified in modifying the three triplet function statemets ((4.2), (4.7) and (4.14)) by adding in the limit statements of Lemmas 4.2, 4.4 and 4.5 to the limit statements of the first, second and third triplet for $\mu(x) \equiv M_{\lambda,\alpha}{}^2(x)$ respectively. Thus given an arbitrary $\epsilon > 0$, we consider first an $a > 0$ such that for the first triplet $\{F_a, F_a{}', F_a{}''\}$ of functions

$$||M_{\lambda,\alpha}[f - F_a]||, \ ||f^{(j)} - F_a{}^{(j)}|| < \epsilon/3 \ (j = 0, 1, 2) \tag{2}$$

holds. Thereafter we choose an $\eta < a/8$ so that for the second triplet $\{(F_a * \delta_\eta), (F_a * \delta_\eta)', (F_a * \delta_\eta)''\}$ of functions

$$||M_{\lambda,\alpha}[F_a - F_a * \delta_\eta]||, \ ||F_a{}^{(j)} - (F_a * \delta_\eta)^{(j)}|| < \epsilon/3 \ (j = 0, 1, 2). \tag{3}$$

becomes valid. And we finally pick an R so large that for the third triplet $\{(F_a * \delta_\eta)\varphi_{R,r}, ((F_a * \delta_\eta)\varphi_{R,r})', ((F_a * \delta_\eta)\varphi_{R,r})''\}$ of functions

$$||M_{\lambda,\alpha}[F_a * \delta_\eta - (F_a * \delta_\eta)\varphi_{R,r}]||, \ ||(F_a * \delta_\eta)^{(j)} - ((F_a * \delta_\eta)\varphi_{R,r})^{(j)}|| < \epsilon/3 \ (j = 0, 1, 2) \tag{4}$$

is guaranteed. By writing $f_\epsilon \equiv (F_a * \delta_\eta)\varphi_{R,r}$ we obtain a $C_0^\infty(0,\infty)$-function for which

$$||M_{\lambda,\alpha}[f - f_\epsilon]||, \ ||f^{(j)} - f_\epsilon{}^{(j)}|| < \epsilon \ (j = 0, 1, 2) \tag{5}$$

and thereby completing our proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

THEOREM 5.2: For all $f \in D(H_\lambda) = D(T_{2,F}) \cap D(M_{\lambda,\alpha})$ and $g \in D(\mathbf{H}_\lambda{}^\dagger)$ we have that

$$< H_\lambda f \mid g > = < f \mid \mathbf{H}_\lambda{}^\dagger g >. \tag{6}$$

PROOF. We apply Theorem 5.1 for an arbitrary $\epsilon > 0$ - i. e. there exists an $f_\epsilon \in C_0^\infty(0,\infty)$ such that $||M_{\lambda,\alpha}[f - f_\epsilon]|| < \epsilon$ and $||f'' - f_\epsilon{}''|| < \epsilon$. We calculate and re-arrange

$$\begin{aligned}
< H_\lambda f \mid g > &= < -f'' + M_{\lambda,\alpha}f \mid g > = < -f'' \mid g > + < M_{\lambda,\alpha}f \mid g > \\
&= < -[f'' - f_\epsilon{}''] - f_\epsilon{}'' \mid g > + < M_{\lambda,\alpha}[f - f_\epsilon] + M_{\lambda,\alpha}f_\epsilon \mid g > \\
&= -< f'' - f_\epsilon{}'' \mid g > + < M_{\lambda,\alpha}[f - f_\epsilon] \mid g > + < \mathbf{H}_\lambda f_\epsilon \mid g > \\
&= -< f'' - f_\epsilon{}'' \mid g > + < M_{\lambda,\alpha}[f - f_\epsilon] \mid g > + < f_\epsilon \mid \mathbf{H}_\lambda{}^\dagger g > \\
&= -< f'' - f_\epsilon{}'' \mid g > + < M_{\lambda,\alpha}[f - f_\epsilon] \mid g > - < f - f_\epsilon \mid \mathbf{H}_\lambda{}^\dagger g > + < f \mid \mathbf{H}_\lambda{}^\dagger g >,
\end{aligned} \tag{7}$$

which chain of equalities implies via the Cauchy-Schwarz inequality that

$$|< H_\lambda f \mid g > - < f \mid \mathbf{H}_\lambda{}^\dagger g >| \le |-< f'' - f_\epsilon{}'' \mid g > + < M_{\lambda,\alpha}[f - f_\epsilon] \mid g >$$

$$-< f - f_\epsilon \mid \mathbf{H}_\lambda{}^\dagger g >| \le |< f'' - f_\epsilon{}'' \mid g >| + |<M_{\lambda,\alpha}[f - f_\epsilon] \mid g >| + |< f - f_\epsilon \mid \mathbf{H}_\lambda{}^\dagger g >| \quad (8)$$

$$\le \|f'' - f_\epsilon{}''\| \times \|g\| + \|M_{\lambda,\alpha}[f - f_\epsilon]\| \times \|g\| + \|f - f_\epsilon\| \times \|\mathbf{H}_\lambda{}^\dagger g\| < \epsilon[2\|g\| + \|\mathbf{H}_\lambda{}^\dagger g\|].$$

However, ϵ was an arbitrary positive number, therefore we conclude that

$$< H_\lambda f \mid g >=< f \mid \mathbf{H}_\lambda{}^\dagger g > \text{ for all } f \in D(H_\lambda) \text{ and } g \in D(\mathbf{H}_\lambda{}^\dagger), \quad (9)$$

thereby ending the proof.   □

THEOREM 5.3: For the perturbed Hamiltonian H_λ, we have that $H_\lambda{}^\dagger = \mathbf{H}_\lambda{}^\dagger$.

PROOF. The inner product statement of Theorem 5.2 implies that H_λ and $\mathbf{H}_\lambda{}^\dagger$ are formally adjoint to each other, and thus $H_\lambda{}^\dagger \supset \mathbf{H}_\lambda{}^\dagger$. On the other hand, out of $H_\lambda \supset \mathbf{H}_\lambda$ shall follow $\mathbf{H}_\lambda{}^\dagger \supset H_\lambda{}^\dagger$, and our proof is thus completed.   □

Having obtained the adjoint $H_\lambda{}^\dagger$ of the perturbed Hamiltonian H_λ, we may now turn our attention to the construction of the Friedrichs extension T_λ of H_λ in the Hilbert space $L_2(0,\infty)$.

6. The Friedrichs Extension of H_λ

Prior to embarking upon the derivation of the Friedrichs extensions of the λ-parameter family H_λ of operators, we must return to the triplets at the beginning of Section 4 ((4.2), (4.7) and (4.14)) and modify these triplets to pairs by dropping the second derivatives. We recall that for every $f \in D(T_{2,F})$, we obtained for each of the three triplets $\{F_a, F_a{}', F_a{}''\}$, $\{(F_a * \delta_\eta), (F_a * \delta_\eta)', (F_a * \delta_\eta)''\}$ and $\{(F_a * \delta_\eta)\varphi_{R,r}, ((F_a * \delta_\eta)\varphi_{R,r})', ((F_a * \delta_\eta)\varphi_{R,r})''\}$ the following limit statments $(j = 0, 1, 2)$:

$$\|f^{(j)} - F_a{}^{(j)}\| \to 0, \ \|F_a{}^{(j)} - (F_a * \delta_\eta)^{(j)}\| \to 0 , \ \|(F_a * \delta_\eta)^{(j)} - ((F_a * \delta_\eta)\varphi_{R,r})^{(j)}\| \to 0$$

for $a \to 0^+, \eta \to 0^+$, and $R \to \infty$ respectively. These limit statements can be simply modified for absolutely continuous $L_2(0,\infty)$-functions f with $f(0) = 0$ and $f' \in L_2(0,\infty)$. If f is such a function, then F_a, defined by $F_a(x) \equiv 0$ for $0 \le x < a$ and $\equiv f(x-a)$ for $x \ge a$, is an absolutely continuous $L_2(0,\infty)$-function with $F_a(0) = 0$ and $F_a{}' \in L_2(0,\infty)$ for all $a > 0$. All of these $\|\cdot\|$-limit statements came about by means of: the Jensen integral inequality for the measure $\delta_\eta(x)dx$ on $(-\eta, \eta)$ combined with the Tonnelli-Hobson Theorem, the Minkowski integral inequality, as well as the integration by parts formula going up to the second derivative. Therefore, we treat absolutely continuous $L_2(0,\infty)$-function f with $f(0) = 0$ and $f' \in L_2(0,\infty)$ likewise, however we go up to the first derivative only. Hence, we obtain analoguous $\|\cdot\|$-limit statements for the three pairs $\{F_a, F_a{}'\}, \{(F_a * \delta_\eta), (F_a * \delta_\eta)'\}$ and $\{(F_a * \delta_\eta)\varphi_{R,r}, ((F_a * \delta_\eta)\varphi_{R,r})'\}$ as before, and these are as follows $(j = 0, 1)$:

$$\|f^{(j)} - F_a{}^{(j)}\| \to 0, \|F_a{}^{(j)} - (F_a * \delta_\eta)^{(j)}\| \to 0, \|(F_a * \delta_\eta)^{(j)} - ((F_a * \delta_\eta)\varphi_{R,r})^{(j)}\| \to 0$$

for $a \to 0^+$, $\eta \to 0^+$ and $R \to \infty$ respectively. At this stage we point to the fact that the function $M_{\lambda,\alpha}(x) = [Bx^2 + Ax^{-2} + \lambda x^{-\alpha}]$, which is continuous on $(0,\infty)$, satisfies

the conditions required by the function $\mu(x)$ appearing in Lemma 4.5. Therefore, if we assume for our absolutely continuous $L_2(0,\infty)$-function f, with $f(0) = 0$ and $f' \in L_2(0,\infty)$, the further property of $\sqrt{M_{\lambda,\alpha}}f \in L_2(0,\infty)$, then we shall have in addition to the immediately preceeding $||\cdot||$-limit statements for the given pairs the following:

$$||\sqrt{M_{\lambda,\alpha}}[f - F_a]||\to 0, ||\sqrt{M_{\lambda,\alpha}}[F_a - F_a * \delta_\eta]|| \to 0, ||\sqrt{M_{\lambda,\alpha}}[F_a * \delta_\eta - (F_a * \delta_\eta)\varphi_{R,r}]|| \to 0,$$

for $a \to 0^+$, $\eta \to 0^+$ and $R \to \infty$ respectively. We have consequently arrived at the following

LEMMA 6.1: If f is an absolutely continuous $L_2(0,\infty)$-function f with $f(0) = 0$ and f' and $\sqrt{M_{\lambda,\alpha}}f \in L_2(0,\infty)$, then for $j = 0$ and 1 we have that

$$||\sqrt{M_{\lambda,\alpha}}[F_a - F_a * \delta_\eta]||, ||f^{(j)} - F_a^{(j)}|| \to 0 \text{ for } a \to 0^+; \tag{1}$$

$$||\sqrt{M_{\lambda,\alpha}}[F_a - F_a * \delta_\eta]||, ||F_a^{(j)} - (F_a * \delta_\eta)^{(j)}|| \to 0 \text{ for } \eta \to 0^+; \tag{2}$$

$$||\sqrt{M_{\lambda,\alpha}}[F_a * \delta_\eta - (F_a * \delta_\eta)\varphi_{R,r}]||, ||(F_a * \delta_\eta)^{(j)} - ((F_a * \delta_\eta)\varphi_{R,r})^{(j)}|| \to 0 \text{ for } R \to \infty. \tag{3}$$

These $||\cdot||$-limit statements permit us to fomulate a theorem analoguous to Theorem 4, namely

THEOREM 6.2: To every $f \in A(0,\infty) \cap L_2(0,\infty)$ with $f(0) = 0$, f' and $\sqrt{M_{\lambda,\alpha}}f \in L_2(0,\infty)$ and pre-assigned $\epsilon > 0$, there exists an $f_\epsilon \in C_0^\infty(0,\infty)$ such that

$$||\sqrt{M_{\lambda,\alpha}}[f - f_\epsilon]||, ||f - f_\epsilon^{(j)}|| < \epsilon \ (j = 0, 1). \tag{4}$$

PROOF. We shall apply Lemma 6.1 by first picking an $a > 0$ satisfying $||\sqrt{M_{\lambda,\alpha}}[F_a - F_a * \delta_\eta]||, ||f^{(j)} - F_a^{(j)}|| < \epsilon/3 \ (j = 0, 1)$. Thus we find an $\eta > 0$ so small that $\eta < a/8$, for which $||\sqrt{M_{\lambda,\alpha}}[F_a - F_a * \delta_\eta]||, ||F_a^{(j)} - (F_a * \delta_\eta)^{(j)}|| < \epsilon/3 \ (j = 0, 1)$. And finally, we choose an R so large that $||\sqrt{M_{\lambda,\alpha}}[F_a * \delta_\eta - ((F_a * \delta_\eta)\varphi_{R,r})]||, ||(F_a * \delta_\eta)^{(j)} - ((F_a * \delta_\eta)\varphi_{R,r})^{(j)}|| < \epsilon/3$ $(j = 0, 1)$. Herewith we define $f_\epsilon \equiv (F_a * \delta_\eta)\varphi_{R,r} \in C_0^\infty(0,\infty)$, which, by the choice $\epsilon/3$, gives the concluding statements of this theorem, thus ending the proof. □

We want to emphasize that in the above theorem, the approximating $C_0^\infty(0,\infty)$-function f_ϵ also belongs to $D(H_\lambda) = D(T_{2,F}) \cap D(M_{\lambda,\alpha})$.

To arrive at the Friedrichs extensions T_λ of the λ-parameter family H_λ of operators, we have to return individually to each of the semi-bounded operators H_λ in the Hilbert space $L_2(0,\infty)$, which is each bounded below by $\gamma = \gamma(A, B, \lambda; \alpha) \equiv \min M_{\lambda,\alpha}((0,\infty))$. They generated a λ-parameter family of semi-bounded sesquilinear forms s_λ on $D(H_\lambda) \times D(H_\lambda)$, having lower bound $\gamma = \gamma(A, B, \lambda; \alpha)$, defined by

$$s_\lambda(f, g) \equiv< H_\lambda f \mid g >=< -f'' \mid g > + < M_{\lambda,\alpha}f \mid g > \text{ for all } f, g \in D(H_\lambda). \tag{5}$$

Herein we may write $< -f'' \mid g >=< f' \mid g' >$ in consequence of

$$< -f'' \mid g > = \lim_{R\to\infty} \int_0^R -f''(x)\overline{g(x)}dx = \lim_{R\to\infty} \left[f'(x)\overline{g'(x)}|_0^R + \int_0^R f'(x)\overline{g'(x)}dx \right]$$

$$= \lim_{R\to\infty} \left[f'(R)\overline{g'(R)} + \int_0^R f'(x)\overline{g'(x)}dx \right] = \int_0^\infty f'(x)\overline{g'(x)}dx =< f' \mid g' >,$$

where we have used the fact that $f, g \in D(H_\lambda) = D(T_{2,F}) \cap D(M_{\lambda,\alpha})$ implies $f^{(j)}(0) = g^{(j)}(0) = 0$ as well as $f^{(j)}(R), g^{(j)}(R) \to 0$ as $R \to \infty$ $(j = 0, 1)$ since $[10, \text{p } 153, \text{Theorem } 6.27]$ $f, g \in W_{2,2}(0, \infty)$. We may therefore replace $< -f'' \mid g >$ with $< f' \mid g' >$ in the definition of s_λ, and thus have that

$$s_\lambda(f, g) \equiv < H_\lambda f \mid g > = < f' \mid g' > + < \sqrt{M_{\lambda,\alpha}} f \mid \sqrt{M_{\lambda,\alpha}} g > \text{ for all } f, g \in D(H_\lambda) \qquad (6)$$

is a semi-bounded sesquilinear form bounded below by $\gamma = \gamma(A, B, \lambda; \alpha)$, which induces the λ-parameter family of inner products

$$< f \mid g >_\lambda \equiv s_\lambda(f, g) + [1 - \gamma] < f \mid g >$$
$$= < f' \mid g' > + < \sqrt{M_{\lambda,\alpha}} f \mid \sqrt{M_{\lambda,\alpha}} g > + [1 - \gamma] < f \mid g >; \qquad (7)$$

on $D(H_\lambda)$, in other words, $(D(H_\lambda), < \cdot \mid \cdot >_\lambda)$ is a Pre-Hilbert space, whose norm

$$||f||_\lambda \equiv \sqrt{||f'||^2 + ||\sqrt{M_{\lambda,\alpha}} f||^2 + [1 - \gamma] ||f||^2} \quad \left(\geq ||f|| \right) \qquad (8)$$

is compatible with the original $L_2(0, \infty)$-norm $|| \cdot ||$. This follows from the fact that each of the sesquilinear forms s_λ was defined $[10, \text{p } 119]$- $[13, \text{p } 329, \text{Sec. } 124]$ by the semi-bounded operator H_λ in the Hilbert space $L_2(0, \infty)$. Because of this, we may consider the completion $(\mathcal{H}_\lambda, < \cdot \mid \cdot >_\lambda)$ of $(D(H_\lambda), < \cdot \mid \cdot >_\lambda)$ as a subspace of $L_2(0, \infty)$. As a result of this, every $f \in \mathcal{H}_\lambda$ must be absolutely continuous with $||f'||^2 + ||\sqrt{M_{\lambda,\alpha}} f||^2 + [1 - \gamma] ||f||^2 < \infty$ and moreover $f(0) = 0$, because any sequence $\{f_n\}$ from $D(H_\lambda)$ converging to $f \in \mathcal{H}_\lambda$ in norm $|| \cdot ||_\lambda$ satisfies

$$f_n(x) = \int_0^x f_n'(t) dt \to \int_0^x f'(t) dt = f(x) \text{ as } n \to \infty.$$

This lets us thoroughly describe the completion $\mathcal{H}_\lambda$ of $D(H_\lambda)$ in terms of the norm $|| \cdot ||_\lambda$ as

$$\mathcal{H}_\lambda = \{ f \in L_2(0, \infty) : ||f'||^2 + ||\sqrt{M_{\lambda,\alpha}} f||^2 + [1 - \gamma] ||f||^2 < \infty \}$$

and these subspaces of $L_2(0, \infty)$ we formulate more elegantly in terms of

THEOREM 6.3: The completions $\mathcal{H}_\lambda$ of $D(H_\lambda)$ in terms of the respective norms $|| \cdot ||_\lambda$ is

$$\{ f \in L_2(0, \infty) : f \in A(0, \infty), f(0) = 0, \text{ both } f' \text{ and } \sqrt{M_{\lambda,\alpha}} f \in L_2(0, \infty) \}. \qquad (9)$$

PROOF. The completion $\mathcal{H}_\lambda$ is clear from the above. Therefore, let us assume that $f \in L_2(0, \infty)$ satisfies all the conditions specified within the curly brackets. As consequence of Theorem 6.2 we can find a sequence of $C_0^\infty(0, \infty)$-functions $\{f_n\}$ such that $||\sqrt{M_{\lambda,\alpha}} [f - f_n]||, ||f - f_n||$ and $||f' - f_n'|| \to 0$ as $n \to \infty$, which sequence also belongs to $D(H_\lambda) = D(T_{2,F}) \cap D(M_{\lambda,\alpha})$. This sequence is a $|| \cdot ||_\lambda$-Cauchy sequence from $D(H_\lambda)$ with the property that

$$||f - f_n||_\lambda = \sqrt{||f' - f_n'||^2 + ||\sqrt{M_{\lambda,\alpha}} [f - f_n]||^2 + [1 - \gamma] ||f - f_n||^2} \to 0 \text{ as } n \to \infty \text{ and}$$

$$f_n(x) = \int_0^x f_n'(t) dt \to \int_0^x f'(t) dt = f(x) \text{ as } n \to \infty \text{ for all } x \geq 0.$$

Thereby the proof is complete.   □

Finally, we come to the construction of the Friedrichs extension T_λ of perturbed Hamiltonian operator H_λ in the Hilbert space $L_2(0,\infty)$. As is well established [10, p 120, Theorem 5.38]- [13, p 335], the operator T_λ has domain of definition $D(T_\lambda) = D(H_\lambda{}^\dagger) \cap \mathcal{H}_\lambda$ and action $T_\lambda f \equiv H_\lambda{}^\dagger f$ for all $f \in D(T_\lambda)$. As a direct result of the immediately preceeding paragraph and Theorem 6.3, we may now formulate

THEOREM 6.4 The Friedrichs extension T_λ of the perturbed Hamiltonian operator H_λ has domain of definition and action given by

$$D(T_\lambda) = \{f \in L_2(0,\infty) : (-f'' + M_{\lambda,\alpha}f) \in L_2(0,\infty), f(0) = 0, f \in A(0,\infty) \text{ with}$$

$$\text{both } f' \text{ and } \sqrt{M_{\lambda,\alpha}}f \in L_2(0,\infty)\} \text{ and } T_\lambda f = H_\lambda{}^\dagger f \text{ for all } f \in D(T_\lambda). \tag{10}$$

Acknowledgments

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